Numerical Solutions of Systems of High-Order Linear Differential-Difference Equations with Bessel Polynomial Bases

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In this paper, a numerical matrix method, which is based on collocation points, is presented for the approximate solution of a system of high-order linear differential-difference equations with variable coefficients under the mixed conditions in terms of Bessel polynomials. Numerical examples are included to demonstrate the validity and applicability of the technique and comparisons are made with existing results. The results show the efficiency and accuracy of the present work. All of the numerical computations have been performed on computer using a program written in MATLAB v7.6.0 (R2008a).

Key words: System of Differential-Difference Equations; Bessel Polynomials and Series; Bessel Polynomial Solutions; Bessel Matrix Method; Collocation Points.

1. Introduction

Systems of high-order linear differential-difference equations and their solutions play a major role in science and engineering. The behaviour of most physical systems, such as biological applications in population dynamics and genetics, can be modelled by systems of differential equations. The mentioned systems of differential-difference equations are usually difficult to solve analytically; so a numerical method is required. Recently, systems of linear differential equations were solved using variational iteration method [1], differential transformation method [2], Adomian decomposition method [3], differential transform method [4], linearizability criteria [5], decomposition method [6], Tau method [7], and homotopy analysis method [8]. In adidition, solutions of systems of linear integral and integro-differential equations were obtained using variational iteration method [9], Taylor matrix method [10], Haar functions method [11, 12], Tau method [13], differential transform method [14], an efficient algorithm [15], and homotopy perturbation method [16]. Also, systems of linear differential and integro-differential equations were solved using the Taylor collocation method and the Chebyshev collocation method by the research group around M. Sezer [17 – 20].

Since the beginning of 1994, Taylor, Chebyshev, and Legendre matrix methods have been used by Sezer et al. [17 – 24] to solve differential, Fredholm–Volterra integral, Fredholm–Volterra integro-differential-difference equations, and systems of linear differential and Fredholm–Volterra integro-differential equations. Also, the Bessel matrix method has been used to find the approximate solutions of differential, integral, and integro-differential equations [25].

In this study, in the light of the above-mentioned methods and by means of the matrix relations between the Bessel polynomials and their derivatives, we develope a new method which is called Bessel matrix method for solving a system of high-order linear differential-difference equations with variable coefficients in the form

$$\sum_{n=0}^{m}\sum_{j=1}^{k} P_{i,j}^{n}(x)y_j^{(n)}(\alpha x+\mu)=g_i(x), \quad i=1,2,\dots,k,\ \ 0\le a\le x\le b \tag{1}$$

with the mixed conditions

$$\sum_{j=0}^{m-1}\left(a_{i,j}^{n}y_n^{(j)}(a)+b_{i,j}^{n}y_n^{(j)}(b)\right)=\lambda_{n,i}, \quad i=0,1,\dots,m-1,\ \ n=1,2,\dots,k, \tag{2}$$

where $y_j^{(0)}(x) = y_j(x)$, and $y_j(x)$ is an unknown function. The known functions $P_{i,j}^n(x)$ and $g_i(x)$ are defined on the interval $a \le x \le b$, and also α, μ, $a_{i,j}^n$, $b_{i,j}^n$, and $\lambda_{n,i}$ are appropriate constants.

Our aim is to find an approximate solution of (1) expressed in the truncated Bessel series form

$$y_i(x) = \sum_{n=0}^{N} a_{i,n} J_n(x), \quad i = 1,2,\dots,k, \ \ 0 \le a \le x \le b, \tag{3}$$

so that $a_{i,n}$, $n = 0,1,2,\dots,N$ are the unknown Bessel coefficients, N is chosen any positive integer such that $N \ge m$, and $J_n(x)$, $n = 0,1,2,\dots,N$ are the Bessel polynomials of first kind defined by

$$J_n(x) = \sum_{k=0}^{\frac{N-n}{2}} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}, \ \ n \in \mathbb{N}, \ \ 0 \le x < \infty.$$

2. Fundamental Matrix Relations

Firstly, we can write $J_n(x)$ in the matrix form as follows:

$$\mathbf{J}^{\mathbf{T}}(x) = \mathbf{D}\mathbf{X}^{\mathbf{T}}(x) \Leftrightarrow \mathbf{J}(x) = \mathbf{X}(x)\mathbf{D}^{\mathbf{T}}, \tag{4}$$

where

$$\mathbf{J}(x) = \begin{bmatrix} J_0(x) & J_1(x) & \cdots & J_N(x) \end{bmatrix} \text{ and}$$
$$\mathbf{X}(x) = \begin{bmatrix} 1 & x^1 & x^2 & \cdots & x^N \end{bmatrix}$$

if N is odd,

$$\mathbf{D} = \begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N-1}{2}\right)!2^{N-1}} & 0 \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N+1}{2}\right)!2^{N}} \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & \frac{(-1)^{\frac{N-3}{2}}}{\left(\frac{N-3}{2}\right)!\left(\frac{N+1}{2}\right)!2^{N-1}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{bmatrix}_{(N+1)\times(N+1)}$$

if N is even,

$$\mathbf{D} = \begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!\left(\frac{N}{2}\right)!2^{N}} \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N}{2}\right)!2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N+2}{2}\right)!2^{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{bmatrix}_{(N+1)\times(N+1)}.$$

We consider the desired solutions $y_j(x)$, $j = 1,2,\dots,k$ of (1) defined by the truncated Bessel series (3). Then the function defined in relation (3) can be written in the matrix form

$$[y_j(x)] = \mathbf{J}(x)\mathbf{A}_j; \ \ \mathbf{A}_j = \begin{bmatrix} a_{j,0} & a_{j,1} & \cdots & a_{j,N} \end{bmatrix}^{\mathbf{T}}, \ \ j = 1,2,\dots,k$$

or from (4)

$$[y_j(x)] = \mathbf{X}(x)\mathbf{D}^{\mathbf{T}}\mathbf{A}_j, \ \ j = 1,2,\dots,k. \tag{5}$$

On the other hand, the relation between the matrix $\mathbf{X}(x)$ and its derivative $\mathbf{X}^{(1)}(x)$ is

$$\mathbf{X}^{(1)}(x) = \mathbf{X}(x)\mathbf{B}^{\mathbf{T}}, \tag{6}$$

where

$$\mathbf{B}^{\mathbf{T}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

From the matrix (6), it follows that

$$\begin{aligned} \mathbf{X}^{(0)}(x) &= \mathbf{X}(x), \\ \mathbf{X}^{(1)}(x) &= \mathbf{X}(x)\mathbf{B}^{\mathbf{T}}, \\ \mathbf{X}^{(2)}(x) &= \mathbf{X}^{(1)}(x)\mathbf{B}^{\mathbf{T}} = \mathbf{X}(x)(\mathbf{B}^{\mathbf{T}})^2, \\ \vdots \quad & \quad \vdots \quad \quad \quad \vdots, \\ \mathbf{X}^{(n)}(x) &= \mathbf{X}^{(n-1)}(x)\mathbf{B}^{\mathbf{T}} = \mathbf{X}(x)(\mathbf{B}^{\mathbf{T}})^n, \end{aligned} \tag{7}$$

where $(\mathbf{B}^{\mathbf{T}})^0 = [\mathbf{I}]_{(N+1)\times(N+1)}$ is the unit matrix.

By using the relations (5), (6), and (7) we have recurrence relations

$$\begin{aligned} & y_j^{(n)}(x) = \mathbf{X}^{(n)}(x)\mathbf{D}^{\mathbf{T}}\mathbf{A}_j = \mathbf{X}(x)(\mathbf{B}^{\mathbf{T}})^n\mathbf{D}^{\mathbf{T}}\mathbf{A}_j, \\ & n = 0,1,2,\ldots,m \text{ and } j = 1,2,\ldots,k. \end{aligned} \tag{8}$$

Similarly, we obtain the matrix relations as follows:

$$\begin{aligned} \mathbf{X}(\alpha x+\mu) &= \mathbf{X}(x)\mathbf{B}(\alpha,\mu), \\ y_j(\alpha x+\mu) &= \mathbf{X}(\alpha x+\mu)\mathbf{D}^{\mathbf{T}}\mathbf{A}_j, \\ y_j^{(n)}(\alpha x+\mu) &= \mathbf{X}(x)\mathbf{B}(\alpha,\mu)(\mathbf{B}^{\mathbf{T}})^n\mathbf{D}^{\mathbf{T}}\mathbf{A}_j, \end{aligned} \tag{9}$$

where for $\alpha \neq 0$, and $\mu \neq 0$,

$$\mathbf{B}(\alpha,\mu) = \begin{bmatrix} \binom{0}{0}(\alpha)^0(\mu)^0 & \binom{1}{0}(\alpha)^0(\mu)^1 & \cdots & \binom{N}{0}(\alpha)^0(\mu)^N \\ 0 & \binom{1}{1}(\alpha)^1(\mu)^0 & \cdots & \binom{N}{1}(\alpha)^1(\mu)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\alpha)^N(\mu)^0 \end{bmatrix}$$

and for $\alpha \neq 0$ and $\mu = 0$,

$$\mathbf{B}(\alpha,0) = \begin{bmatrix} (\alpha)^0 & 0 & \cdots & 0 \\ 0 & (\alpha)^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\alpha)^N \end{bmatrix}.$$

Hence, the matrices $\mathbf{y}^{(n)}(x)$, $n = 0,1,2,\ldots,m$ can be expressed as follows:

$$\mathbf{y}^{(n)}(x) = \bar{\mathbf{X}}(x)\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^n\bar{\mathbf{D}}\mathbf{A}, \quad n = 0,1,2,\ldots,m, \tag{10}$$

where

$$\mathbf{y}^{(n)}(x) = \begin{bmatrix} y_1^{(n)}(\alpha x+\mu) \\ y_2^{(n)}(\alpha x+\mu) \\ \vdots \\ y_k^{(n)}\alpha x+\mu) \end{bmatrix},$$

$$\bar{\mathbf{X}}(x) = \begin{bmatrix} \mathbf{X}(x) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(x) \end{bmatrix}_{k\times k},$$

$$\bar{\mathbf{D}} = \begin{bmatrix} \mathbf{D}^{\mathbf{T}} & 0 & \cdots & 0 \\ 0 & \mathbf{D}^{\mathbf{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}^{\mathbf{T}} \end{bmatrix},$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}(\alpha,\mu) & 0 & \cdots & 0 \\ 0 & \mathbf{B}(\alpha,\mu) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}(\alpha,\mu) \end{bmatrix}_{k\times k},$$

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^{\mathbf{T}} & 0 & \cdots & 0 \\ 0 & \mathbf{B}^{\mathbf{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}^{\mathbf{T}} \end{bmatrix}_{k\times k} \text{ and } \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_k \end{bmatrix}.$$

3. Method of Solution

First, we can write the system (1) in the matrix form

$$\sum_{n=0}^{m} \mathbf{P}_n(x)\mathbf{y}^{(n)}(x) = \mathbf{g}(x), \tag{11}$$

where

$$\mathbf{P}_n(x) = \begin{bmatrix} p^n_{1,1}(x) & p^n_{1,2}(x) & \cdots & p^n_{1,k}(x) \\ p^n_{2,1}(x) & p^n_{2,2}(x) & \cdots & p^n_{2,k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p^n_{k,1}(x) & p^n_{k,2}(x) & \cdots & p^n_{k,k}(x) \end{bmatrix},$$

$$\mathbf{y}^{(n)}(x) = \begin{bmatrix} y_1^{(n)}(\alpha x+\mu) \\ y_2^{(n)}(\alpha x+\mu) \\ \vdots \\ y_k^{(n)}\alpha x+\mu) \end{bmatrix} \quad \text{and} \quad \mathbf{g}(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_k(x) \end{bmatrix}.$$

By using (11) and the collocation points defined by

$$x_s = a + \frac{b-a}{N}s, \quad s = 0,1,\ldots,N, \tag{12}$$

the system of the matrix equations is obtained as

$$\sum_{n=0}^{m} \mathbf{P}_n(x_s)\mathbf{y}^{(n)}(x_s) = \mathbf{g}(x_s)$$

or briefly the fundamental matrix equation

$$\sum_{n=0}^{m} \mathbf{P}_n\mathbf{Y}^{(n)} = \mathbf{G}, \tag{13}$$

where

$$\mathbf{P}_n = \begin{bmatrix} \mathbf{P}_n(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_n(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_n(x_N) \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}(x_0) \\ \mathbf{g}(x_1) \\ \vdots \\ \mathbf{g}(x_N) \end{bmatrix}, \quad \mathbf{Y}^{(n)} = \begin{bmatrix} \mathbf{y}^{(n)}(x_0) \\ \mathbf{y}^{(n)}(x_1) \\ \vdots \\ \mathbf{y}^{(n)}(x_N) \end{bmatrix}.$$

Using relation (10) and the collocation points (12), we obtain

$$\mathbf{y}^{(n)}(x_s) = \bar{\mathbf{X}}(x_s)\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^n\bar{\mathbf{D}}\mathbf{A}, \quad s = 0,1,\ldots,N,$$

which can be written as

$$\mathbf{Y}^{(n)} = \mathbf{X}\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^n\bar{\mathbf{D}}\mathbf{A}, \tag{14}$$

where

$$\mathbf{X} = \begin{bmatrix} \bar{\mathbf{X}}(x_0) \\ \bar{\mathbf{X}}(x_1) \\ \vdots \\ \bar{\mathbf{X}}(x_N) \end{bmatrix} \quad \text{and}$$

$$\bar{\mathbf{X}}(x_s) = \begin{bmatrix} \mathbf{X}(x_s) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(x_s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(x_s) \end{bmatrix}_{k\times k},$$

$$s = 0,1,\ldots,N.$$

Substituting relations (14) into (13), we have the fundamental matrix equation

$$\left\{\sum_{n=0}^{m} \mathbf{P}_n\mathbf{X}\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^n\bar{\mathbf{D}}\right\}\mathbf{A} = \mathbf{G}. \tag{15}$$

When the matrices $\mathbf{P}_n$, $\mathbf{X}$, $\tilde{\mathbf{B}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{D}}$, $\mathbf{A}$, and $\mathbf{G}$ in (14) are written in full, it can be seen that their dimensions are, respectively, $k(N+1)\times k(N+1)$, $k(N+1)\times k(N+1)$, $k(N+1)\times k(N+1)$, $k(N+1)\times k(N+1)$, $k(N+1)\times k(N+1)$, $k(N+1)\times 1$, $k(N+1)\times 1$.

Hence, the fundamental matrix equation (15) corresponding to (1) can be written in the form

$$\mathbf{W}\mathbf{A} = \mathbf{G} \quad \text{or} \quad [\mathbf{W};\mathbf{G}], \tag{16}$$

which corresponds to a linear system of $k(N+1)$ algebraic equations in $k(N+1)$ unknown Bessel coefficients so that

$$\mathbf{W} = \sum_{n=0}^{m} \mathbf{P}_n\mathbf{X}\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^n\bar{\mathbf{D}} = [w_{p,q}],$$
$$p,q = 1,2,\ldots,k(N+1).$$

Now let us form the matrix representation of the conditions. Using the conditions (2), we have

$$\sum_{j=0}^{m-1}\left[a^1_{i,j}y_1^{(j)}(a) + b^1_{i,j}y_1^{(j)}(b)\right] = \lambda_{1,i},$$
$$\sum_{j=0}^{m-1}\left[a^2_{i,j}y_2^{(j)}(a) + b^2_{i,j}y_2^{(j)}(b)\right] = \lambda_{2,i},$$
$$\vdots$$
$$\sum_{j=0}^{m-1}\left[a^k_{i,j}y_k^{(j)}(a) + b^k_{i,j}y_k^{(j)}(b)\right] = \lambda_{k,i}$$

or

$$\sum_{j=0}^{m-1}\left[a_j^1 y_1^{(j)}(a)+b_j^1 y_1^{(j)}(b)\right]=\lambda_1,$$
$$\sum_{j=0}^{m-1}\left[a_j^2 y_2^{(j)}(a)+b_j^2 y_2^{(j)}(b)\right]=\lambda_2,$$
$$\vdots$$
$$\sum_{j=0}^{m-1}\left[a_j^k y_k^{(j)}(a)+b_j^k y_k^{(j)}(b)\right]=\lambda_k,$$

where

$$\lambda_i=\begin{bmatrix}\lambda_{i,0}\\ \lambda_{i,1}\\ \vdots\\ \lambda_{i,m-1}\end{bmatrix}_{m\times 1},\quad a_j^i=\begin{bmatrix}a_{0,j}^i\\ a_{1,j}^i\\ \vdots\\ a_{m-1,j}^i\end{bmatrix}_{m\times 1},$$

$$b_j^i=\begin{bmatrix}b_{0,j}^i\\ b_{1,j}^i\\ \vdots\\ b_{m-1,j}^i\end{bmatrix}_{m\times 1},\quad i=1,2,\ldots,k,$$

or briefly

$$\sum_{j=0}^{m-1}\left[a_j\mathbf{y}^{(j)}(a)+b_j\mathbf{y}^{(j)}(b)\right]=\lambda, \tag{17}$$

where

$$a_j=\begin{bmatrix}a_j^1 & 0 & \cdots & 0\\ 0 & a_j^2 & \vdots & 0\\ 0 & 0 & \ddots & \vdots\\ 0 & 0 & \cdots & a_j^k\end{bmatrix}_{k\times k},$$

$$b_j=\begin{bmatrix}b_j^1 & 0 & \cdots & 0\\ 0 & b_j^2 & \vdots & 0\\ 0 & 0 & \ddots & \vdots\\ 0 & 0 & \cdots & b_j^k\end{bmatrix}_{k\times k}\quad\text{and}\quad \lambda=\begin{bmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_k\end{bmatrix}_{k\times 1}.$$

By using (10) at points a and b, substituting the matrices $\mathbf{y}^{(j)}(a)$ and $\mathbf{y}^{(j)}(b)$, which depend on the matrix of the Bessel coefficients $\mathbf{A}$, into (17) and simplifying the result, we obtain

$$\sum_{j=0}^{m-1}\left[a_j\bar{\mathbf{X}}(a)+b_j\bar{\mathbf{X}}(b)\right]\left(\bar{\mathbf{B}}\right)^j\bar{\mathbf{D}}\mathbf{A}=\lambda. \tag{18}$$

Let us define $\mathbf{U}$ as

$$\mathbf{U}=\sum_{j=0}^{m-1}\left[a_j\bar{\mathbf{X}}(a)+b_j\bar{\mathbf{X}}(b)\right]\left(\bar{\mathbf{B}}\right)^j\bar{\mathbf{D}},$$

thus, the fundamental matrix form for conditions becomes

$$\mathbf{UA}=\lambda \quad\text{or}\quad [\mathbf{U};\lambda]. \tag{19}$$

Consequently, by replacing the rows of the matrix $\mathbf{U}$ and λ, by the rows of the matrix $\mathbf{W}$ and $\mathbf{G}$, respectively, we have

$$\tilde{\mathbf{W}}\mathbf{A}=\tilde{\mathbf{G}}. \tag{20}$$

For convenience, if last mk rows of the matrix $\mathbf{W}$ are replaced, the augmented matrix of the above system is as follows [17–20]:

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=\begin{bmatrix}
w_{1,1} & w_{1,2} & \cdots & w_{1,k(N+1)} & ; & g_1(x_0)\\
w_{2,1} & w_{2,2} & \cdots & w_{2,k(N+1)} & ; & g_2(x_0)\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
w_{k,1} & w_{k,2} & \cdots & w_{k,k(N+1)} & ; & g_k(x_0)\\
w_{k+1,1} & w_{k+1,2} & \cdots & w_{k+1,k(N+1)} & ; & g_1(x_1)\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
w_{k(N-m+1),1} & w_{k(N-m+1),2} & \cdots & w_{k(N-m+1),k(N+1)} & ; & g_k(x_{N-m})\\
v_{1,1} & v_{1,2} & \cdots & v_{1,k(N+1)} & ; & \lambda_{1,0}\\
v_{2,1} & v_{2,2} & \cdots & w_{2,k(N+1)} & ; & \lambda_{1,1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
v_{k,1} & v_{k,2} & \cdots & v_{k,k(N+1)} & ; & \lambda_{1,m-1}\\
v_{k+1,1} & v_{k+1,2} & \cdots & v_{k+1,k(N+1)} & ; & \lambda_{2,0}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
v_{mk,1} & v_{mk,2} & \cdots & v_{mk,k(N+1)} & ; & \lambda_{k,m-1}
\end{bmatrix}. \tag{21}$$

However, we do not have to replace the last rows. For example, if the matrix $\mathbf{W}$ is singular, then the rows that have the same factor or all zero are replaced.

If rank $\tilde{\mathbf{W}}$ = rank $[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] = k(N+1)$, then we can write

$$\mathbf{A} = (\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{G}}. \tag{22}$$

Thus, the matrix $\mathbf{A}$ (thereby the unknown Bessel coefficients) is uniquely determined. Also the system (1) with conditions (2) has a unique solution. This solution is given by truncated Bessel series solution (3). On the other hand, when $\left|\tilde{\mathbf{W}}\right| = 0$, if rank $\tilde{\mathbf{W}}$ = rank $[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] < k(N+1)$, then we may find a particular solution. Otherwise if rank $\tilde{\mathbf{W}} \neq$ rank $[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] < k(N+1)$, then it is not a solution.

The number of the conditions must be at most mk to have the unique solution, since the differential-difference equation system has mth-order derivative at most and k unknown functions. On the order hand, the unique solution can sometimes be obtained using fewer conditions to the problem.

4. Accuracy of Solution

We can easily check the accuracy of the method. Since the truncated Bessel series (3) is the approximate solution of (1), when the function $y_{i,N}(x)$, $i = 1,2,\ldots,k$, and its derivatives, are substituted in (1), the resulting equation must be satisfied approximately; that is, for

$$x = x_q \in [a,b], \quad q = 0,1,2,\ldots,$$
$$E_i(x_q) = \left|\sum_{n=0}^{m}\sum_{j=1}^{k} P_{i,j}^{n}(x_q) y_j^{(n)}(\alpha x_q + \mu) = g_i(x_q)\right| \cong 0,$$
$$i = 1,2,\ldots,k, \tag{23}$$

and $E_i(x_q) \leq 10^{-k_q}$ (k_q positive integer).

If $\max\ 10^{-k_q} = 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E_i(x_q)$ at each of the points becomes smaller than the prescribed 10^{-k}, see [10, 11, 22, 23]. On the other hand, the error can be estimated by the function

$$E_{i,N}(x) = \sum_{n=0}^{m}\sum_{j=1}^{k} P_{i,j}^{n}(x) y_{j,N}^{(n)}(\alpha x + \mu) = g_i(x) \cong 0,$$
$$i = 1,2,\ldots,k.$$

5. Numerical Examples

In this section, several numerical examples are given to illustrate the accuracy and effectiveness properties of the method and all of them were performed on the computer using a program written in MATLAB v7.6.0 (R2008a). In this regard, we have reported in tables and figures, the values of the exact solutions $y_i(x)$, $i = 1,2,\ldots,k$, the polynomial approximate solutions $y_{i,N}(x)$, $i = 1,2,\ldots,k$, and the absolute error functions $e_{i,N}(x) = |y_i(x) - y_{i,N}(x)|$, $i = 1,2,\ldots,k$, at the selected points of the given interval.

Example 1. [26] Let us consider first the linear differential-difference system given by

$$\begin{cases} y_1^{(2)}(2x-1) + xy_2^{(1)}(2x-1) + xy_1(2x-1) = \\ x^3 + 11x^2 - 7x + 2 \\ y_2^{(2)}(2x-1) + 2y_1^{(1)}(2x-1) - 2xy_2(2x-1) = \\ -6x^3 + 4x^2 + 10x \end{cases},$$
$$0 \leq x \leq 1, \tag{24}$$

with the boundary conditions $y_1(0) = y_1(1) = 1$, $y_2(0) = -1$, $y_2(1) = 0$ and approximate the solution $y(x)$ by the truncated Bessel series

$$y_i(x) = \sum_{n=0}^{2} a_{i,n} J_n(x), \quad i = 1,2,$$

where

$$N = 2, k = 2, m = 2, \alpha = 2, \mu = 2,$$
$$g_1(x) = x^3 + 11x^2 - 7x + 2, g_2(x) = -6x^3 + 4x^2 + 10x,$$
$$p_{1,1}^0(x) = x, p_{1,2}^0(x) = 0, p_{2,1}^0(x) = 0, p_{2,2}^0(x) = -2x,$$
$$p_{1,1}^1(x) = 0, p_{1,2}^1(x) = x, p_{2,1}^1(x) = 2, p_{2,2}^1(x) = 0,$$
$$p_{1,1}^2(x) = 1, p_{1,2}^2(x) = 0, p_{2,1}^2(x) = 0 \text{ and } p_{2,2}^2(x) = 1.$$

Hence, the set of collocation points (12) for $N = 2$ is computed as

$$\left\{x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1\right\}$$

and from (15), the fundamental matrix equation of the problem is

$$\left\{\mathbf{P}_0\mathbf{X}\tilde{\mathbf{B}}\bar{\mathbf{D}} + \mathbf{P}_1\mathbf{X}\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^1\bar{\mathbf{D}} + \mathbf{P}_2\mathbf{X}\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^2\bar{\mathbf{D}}\right\}\mathbf{A} = \mathbf{G},$$

where

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{P}_0(0) & 0 & 0 \\ 0 & \mathbf{P}_0(1/2) & 0 \\ 0 & 0 & \mathbf{P}_0(1) \end{bmatrix},$$

$$\mathbf{P}_1=\begin{bmatrix}\mathbf{P}_1(0) & 0 & 0\\ 0 & \mathbf{P}_1(1/2) & 0\\ 0 & 0 & \mathbf{P}_1(1)\end{bmatrix},$$
$$\mathbf{P}_2=\begin{bmatrix}\mathbf{P}_2(0) & 0 & 0\\ 0 & \mathbf{P}_2(1/2) & 0\\ 0 & 0 & \mathbf{P}_2(1)\end{bmatrix},$$
$$\mathbf{X}=\begin{bmatrix}\bar{\mathbf{X}}(0)\\ \bar{\mathbf{X}}(1/2)\\ \bar{\mathbf{X}}(1)\end{bmatrix},\ \bar{\mathbf{X}}(0)=\begin{bmatrix}\mathbf{X}(0) & 0\\ 0 & \mathbf{X}(0)\end{bmatrix},$$
$$\bar{\mathbf{X}}(1/2)=\begin{bmatrix}\mathbf{X}(1/2) & 0\\ 0 & \mathbf{X}(1/2)\end{bmatrix},$$
$$\bar{\mathbf{X}}(1)=\begin{bmatrix}\mathbf{X}(1) & 0\\ 0 & \mathbf{X}(1)\end{bmatrix},\ \mathbf{X}(0)=\begin{bmatrix}1 & 0 & 0\end{bmatrix},$$
$$\mathbf{X}(1/2)=\begin{bmatrix}1 & 1/2 & 1/4\end{bmatrix},\ \mathbf{X}(1)=\begin{bmatrix}1 & 1 & 1\end{bmatrix},$$

$$\tilde{\mathbf{B}}=\begin{bmatrix}\mathbf{B}(\alpha,\mu) & 0\\ 0 & \mathbf{B}(\alpha,\mu)\end{bmatrix},\ \bar{\mathbf{B}}=\begin{bmatrix}\mathbf{B^T} & 0\\ 0 & \mathbf{B^T}\end{bmatrix},$$
$$\mathbf{B}(\alpha,\mu)=\mathbf{B}(2,-1)=\begin{bmatrix}1 & -1 & 1\\ 0 & 2 & -4\\ 0 & 0 & 4\end{bmatrix},$$
$$\bar{\mathbf{D}}=\begin{bmatrix}\mathbf{D^T} & 0\\ 0 & \mathbf{D^T}\end{bmatrix},\ \mathbf{B^T}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix},$$
$$\mathbf{D^T}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2 & 0\\ -1/4 & 0 & 1/8\end{bmatrix},$$
$$\mathbf{G}=\begin{bmatrix}\mathbf{g}(0)\\ \mathbf{g}(1/2)\\ \mathbf{g}(1)\end{bmatrix},\ \mathbf{g}(0)=\begin{bmatrix}2\\ 0\end{bmatrix},$$
$$\mathbf{g}(1/2)=\begin{bmatrix}11/8\\ 21/4\end{bmatrix},\ \mathbf{g}(1)=\begin{bmatrix}1\\ 8\end{bmatrix},$$
$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\end{bmatrix},\ \mathbf{A}_1=\begin{bmatrix}a_{1,0}\\ a_{1,1}\\ a_{1,2}\end{bmatrix}\ \text{and}\ \mathbf{A}_2=\begin{bmatrix}a_{2,0}\\ a_{2,1}\\ a_{2,2}\end{bmatrix}.$$

The augmented matrix for this fundamental matrix equation is

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix}-1/2 & 0 & 1/4 & 0 & 0 & 0 & ; & 2\\ 1 & 1 & -1/2 & -1/2 & 0 & 1/4 & ; & 0\\ 0 & 0 & 1/4 & 0 & 1/4 & 0 & ; & 11/8\\ 0 & 1 & 0 & 3/2 & 0 & 1/4 & ; & 21/4\\ 1/4 & 1/2 & 3/8 & -1/2 & 1/2 & 1/4 & ; & 7\\ -1 & 1 & 1/2 & -2 & -1 & 0 & ; & 8\end{bmatrix}.$$

From (19), the matrix form for boundary conditions is

$$[\mathbf{U};\lambda]=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & ; & 0\\ 3/4 & 1/2 & 1/8 & 0 & 0 & 0 & ; & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & ; & 0\\ 0 & 0 & 0 & 3/4 & 1/2 & 1/8 & ; & 0\end{bmatrix}.$$

Hence, from system (21), the new augmented matrix based on conditions can be obtained as

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=\begin{bmatrix}-1/2 & 0 & 1/4 & 0 & 0 & 0 & ; & 2\\ 1 & 1 & -1/2 & -1/2 & 0 & 1/4 & ; & -2\\ 1 & 0 & 0 & 0 & 0 & 0 & ; & 0\\ 3/4 & 1/2 & 1/8 & 0 & 0 & 0 & ; & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & ; & 0\\ 0 & 0 & 0 & 3/4 & 1/2 & 1/8 & ; & 0\end{bmatrix}.$$

By solving this system, the unknown Bessel coefficients matrix is obtained as

$$A=\begin{bmatrix}0 & -2 & 10 & -1 & -4 & 22\end{bmatrix}^T.$$

Substituting the elements of the column matrix into (4), we have $y_1(x)=x^2-x+1$, $y_2(x)=3x^2-2x-1$ which are the exact solutions of (24).

Example 2. [4] Consider the following linear differential system:

$$\begin{aligned}&y_1^{(1)}(x)+y_2^{(1)}(x)+y_1(x)+y_2(x)=1,\\ &y_2^{(1)}(x)-2y_1(x)-y_2(x)=0,\\ &0\le x\le 1,\end{aligned}\tag{25}$$

with the initial conditions $y_1(0)=0$, $y_2(0)=1$, and the exact solutions $y_1(x)=\mathrm{e}^{-x}-1$, $y_2(x)=2-\mathrm{e}^{-x}$. Here, $k=2, m=1, \alpha=1, \mu=0, g_1(x)=1, g_2(x)=0,$
$p^0_{1,1}(x)=1, p^0_{1,2}(x)=1, p^0_{2,1}(x)=-2, p^0_{2,2}(x)=-1,$
$p^1_{1,1}(x)=1, p^1_{1,2}(x)=1, p^1_{2,1}(x)=0,$ and $p^1_{2,2}(x)=1.$

From (15), the fundamental matrix equation of the problem is

$$\left\{\mathbf{P}_0\mathbf{X}\tilde{\mathbf{B}}\bar{\mathbf{D}}+\mathbf{P}_1\mathbf{X}\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^1\bar{\mathbf{D}}\right\}\mathbf{A}=\mathbf{G}.$$

As Example 1, we obtain the approximate solutions by Bessel polynomials of the problem for $N=3$, 6, 10. In Tables 1–4 and Figure 1 the numerical results obtained by the present method for $N=3,6,10$ are compared with the differential transform method [4] of (25). The current method seems more rapidly convergent than the differential transform method in [4].

Table 1. Numerical results of solutions $y_1(x)$ of (25).

x_i	Exact solution $y_1(x_i) = e^{-x_i} - 1$	Transform method [4] $N=6$, $y_{1,6}(x_i)$	Present method $N=3$, $y_{1,3}(x_i)$	$N=6$, $y_{1,6}(x_i)$	$N=10$, $y_{1,10}(x_i)$
0	0	0	0	0	0
0.2	−0.181269246922018	−0.165693777778	−0.181513513514	−0.181269275382	−0.181269246922045
0.4	−0.329679953964361	−0.278471111111	−0.329945945946	−0.329679971784	−0.329679953964381
0.6	−0.451188363905974	−0.349692000000	−0.451135135135	−0.451188376574	−0.451188363906005
0.8	−0.550671035882778	−0.384369777778	−0.550918918919	−0.550671069420	−0.550671035882850
1	−0.632120558828558	−0.398611111111	−0.635135135136	−0.632119872254	−0.632120558827007

Table 2. Numerical results of solutions $y_2(x)$ of (25).

x_i	Exact solution $y_2(x_i) = 2 - e^{-x_i}$	Transform method [4] $N=6$, $y_{2,6}(x_i)$	Present method $N=3$, $y_{2,3}(x_i)$	$N=6$, $y_{2,6}(x_i)$	$N=10$, $y_{2,10}(x_i)$
0	1	1	1	1	1
0.2	1.18126924692202	1.18359546667	1.18151351351	1.18126927538	1.18126924692205
0.4	1.32967995396436	1.34654720000	1.32994594595	1.32967997178	1.32967995396440
0.6	1.45118836390597	1.50568720000	1.45113513514	1.45118837657	1.45118836390605
0.8	1.55067103588278	1.68261546667	1.55091891892	1.55067106942	1.55067103588294
1	1.63212055882856	1.91250000000	1.63513513514	1.63211987225	1.63212055882717

Table 3. Numerical results of the absolute error functions $e_{1,N}(x)$ for $N = 3, 6, 10$ of $y_1(x)$ of (25).

x_i	Transform method [4] $e_{1,6}(x_i)$	Present method $e_{1,3}(x_i)$	$e_{1,6}(x_i)$	$e_{1,10}(x_i)$
0	0	0	0	0
0.2	1.5575e-002	2.4427e-004	2.8460e-008	2.6645e-014
0.4	5.1209e-002	2.6599e-004	1.7820e-008	2.0650e-014
0.6	1.0150e-001	5.3229e-005	1.2668e-008	3.1530e-014
0.8	1.6630e-001	2.4788e-004	3.3538e-008	7.1498e-014
1	2.3351e-001	3.0146e-003	6.8657e-007	1.5510e-012

Table 4. Numerical results of the absolute error functions $e_{2,N}(x)$ for $N = 3, 6, 10$ of $y_2(x)$ of (25).

x_i	Transform method [4] $e_{2,6}(x_i)$	Present method $e_{2,3}(x_i)$	$e_{2,6}(x_i)$	$e_{2,10}(x_i)$
0	0	0	0	0
0.2	2.3262e-003	2.4427e-004	2.8460e-008	2.9421e-014
0.4	1.6867e-002	2.6599e-004	1.7820e-008	3.6082e-014
0.6	5.4499e-002	5.3229e-005	1.2668e-008	7.4274e-014
0.8	1.3194e-001	2.4788e-004	3.3538e-008	1.6065e-013
1	2.8038e-001	3.0146e-003	6.8657e-007	1.3884e-012

As can be seen from Tables 1, 2, and Figure 1 (a)–(b), the results obtained by the Bessel polynomial method for $N = 10$ are as good as the results of the exact solutions.

Example 3. [19] Let us consider the initial value problem

$$\begin{aligned} &y_1^{(1)}(x) + y_2^{(1)}(x) + y_2(x) = x - e^{-x}, \\ &y_1^{(1)}(x) + 4y_2^{(1)}(x) + y_1(x) = 1 + 2e^{-x}, \\ &0 \le x \le 1, \end{aligned} \tag{26}$$

with the initial conditions $y_1(0) = 1$, $y_2(0) = 0$, and the exact solutions $y_1(x) = e^{-x} + 3e^{-x/3} - 3$, $y_2(x) = -(1/2)e^{-x} + (3/2)e^{-x/3} - 1 + x$.

From (15), the fundamental matrix equation of the problem is

$$\left\{ \mathbf{P}_0 \mathbf{X} \tilde{\mathbf{B}} \bar{\mathbf{D}} + \mathbf{P}_1 \mathbf{X} \tilde{\mathbf{B}} \left(\bar{\mathbf{B}} \right)^1 \bar{\mathbf{D}} \right\} \mathbf{A} = \mathbf{G}.$$

Using the procedure in Section 3, the approximate solutions by Bessel polynomials of the problem for $i = 1, 2$ and $N = 5, 7, 10$ become, respectively,

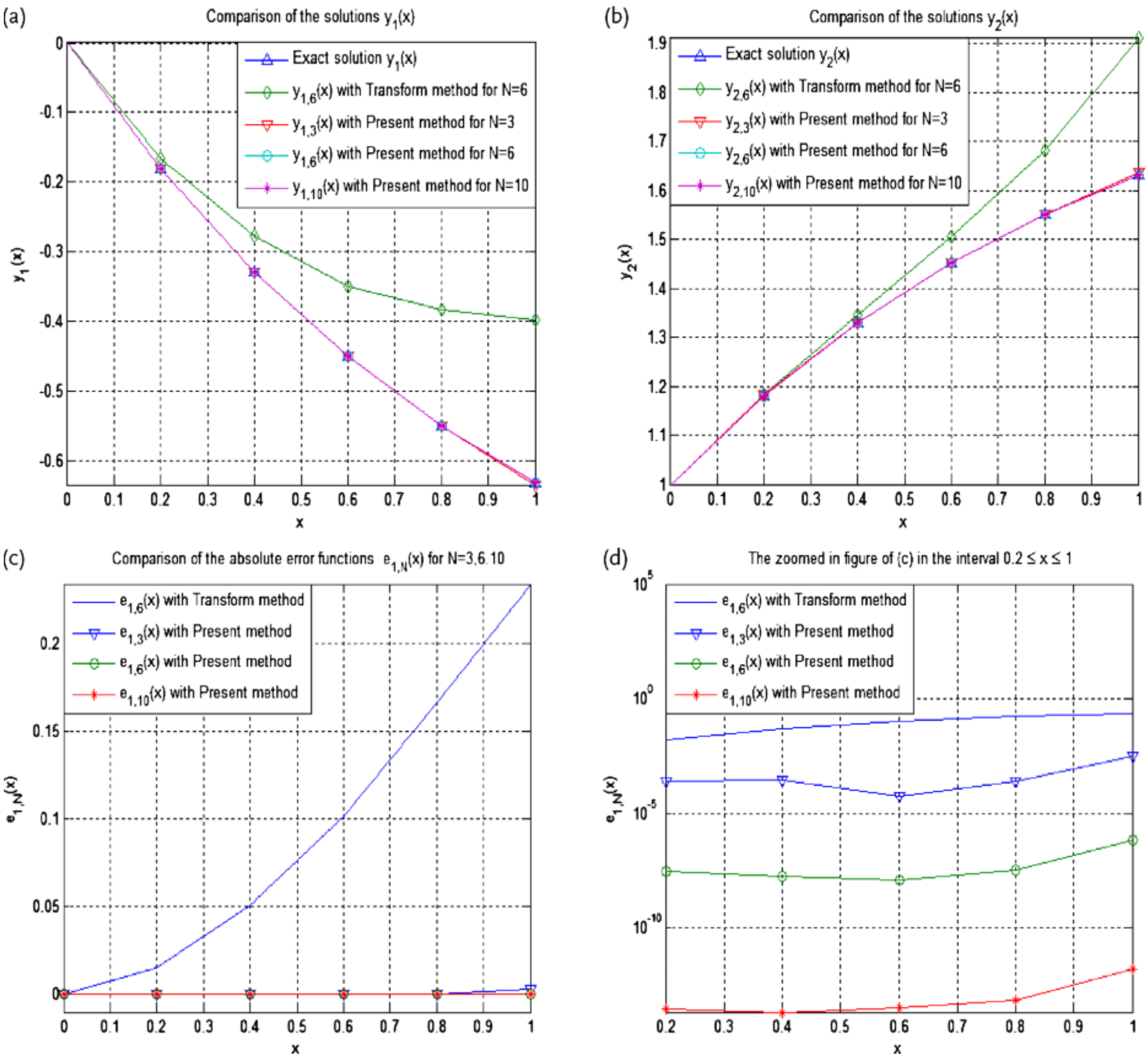

Fig. 1 (colour online). For $N=3$, $N=6$, and $N=10$ of (25), (a) comparison of the exact solution $y_1(x)$ and the approximate solutions $y_{1,N}(x)$, (b) comparison of the exact solution $y_2(x)$ and the approximate solutions $y_{2,N}(x)$, (c) comparison of the absolute errors functions $e_{i,N}(x)$ for $y_{1,N}(x)$, (d) zoomed region in Figure (c) in the interval $0.2 \le x \le 1$, (e) comparison of the absolute errors functions $e_{i,N}(x)$ for $y_{2,N}(x)$, and (f) zoomed region in Figure (e) in the interval $0.2 \le x \le 1$.

$$
\begin{aligned}
y_{1,5}(x) &= 1 - 2x + 0.666542960004x^2 \\
&- 0.184340222499x^3 + (0.409518052791\mathrm{e}-1)x^4 \\
&- (0.569482927687\mathrm{e}-2)x^5, \\
y_{2,5}(x) &= 0.322356379829\mathrm{e}-15 + x \\
&- 0.166605390222x^2 + (0.736556629851\mathrm{e}-1)x^3 \\
&- (0.189435481094\mathrm{e}-1)x^4 \\
&+ (0.275727613667\mathrm{e}-2)x^5,
\end{aligned}
$$

$$
\begin{aligned}
y_{1,7}(x) &= 1 - 2x + 0.666666224495x^2 \\
&- 0.185180204120x^3 + (0.431857426738\mathrm{e}-1)x^4 \\
&- (0.837418984174\mathrm{e}-2)x^5 \\
&+ (0.130569691295\mathrm{e}-2)x^6 \\
&- (0.129930436938\mathrm{e}-3)x^7, \\
y_{2,7}(x) &= (0.589798731458\mathrm{e}-15) + x \\
&- 0.166666445821x^2 + (0.740715863216\mathrm{e}-1)x^3
\end{aligned}
$$

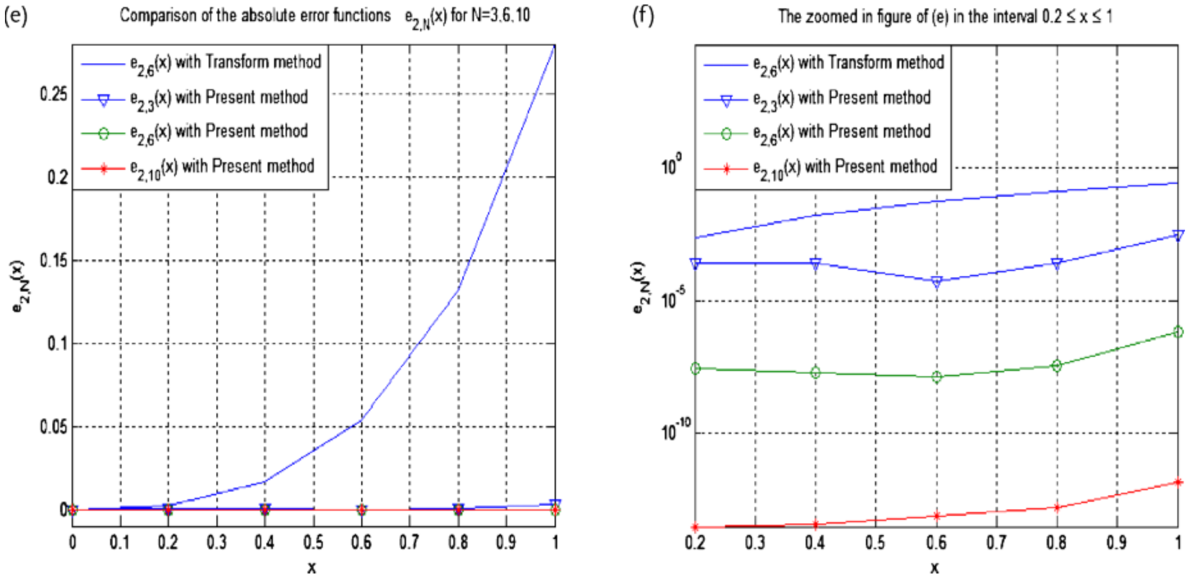

Fig. 1 (colour online). Continued.

$$
\begin{aligned}
&-(0.200496750005\mathrm{e}-1)x^4\\
&+(0.408424889553\mathrm{e}-2)x^5\\
&-(0.647181636581\mathrm{e}-3)x^6\\
&+(0.647291008150\mathrm{e}-4)x^7,
\end{aligned}
$$

and

$$
\begin{aligned}
y_{1,10}(x) &= 1-2x+0.666666666632x^2\\
&-0.185185184541x^3+(0.432098710323\mathrm{e}-1)x^4\\
&-(0.843618669765\mathrm{e}-2)x^5\\
&+(0.139451921253\mathrm{e}-2)x^6\\
&-(0.198510974616\mathrm{e}-3)x^7\\
&+(0.245807788702\mathrm{e}-4)x^8\\
&-(0.255882907561\mathrm{e}-5)x^9\\
&+(0.176281080549\mathrm{e}-6)x^{10},\\
y_{2,10}(x) &= x-0.166666666649x^2\\
&+(0.740740737519\mathrm{e}-1)x^3\\
&-(0.200617256374\mathrm{e}-1)x^4\\
&+(0.411521267955\mathrm{e}-2)x^5\\
&-(0.691543983619\mathrm{e}-3)x^6\\
&+(0.989832649404\mathrm{e}-4)x^7\\
&-(0122789983698\mathrm{e}-4)x^8\\
&+(0.127896801267\mathrm{e}-5)x^9\\
&-(0.881213987119\mathrm{e}-7)x^{10}.
\end{aligned}
$$

Tables 5–6 and Figure 2 display the comparison of results of the absolute error functions obtained by the Stehfest method [27], the Chebyshev method [19], and the present method for $N = 5,7,10$ of the system in (26). It is seen from Tables 5–6 and Figure 2 that the results obtained by the present method are better than those obtained by the other methods. Tables 5–6 and Figure 2 show that as N increases, the errors decrease more rapidly.

Example 4. Finaly, consider the linear fourth-order differential-difference system with variable coefficients

$$
\begin{aligned}
&y_1^{(4)}(0.5x-0.3)-2y_2^{(3)}(0.5x-0.3)\\
&-y_3^{(1)}(0.5x-0.3)+y_1(0.5x-0.3) = -\mathrm{e}^{\left(\frac{1}{2}x-\frac{3}{10}\right)},\\
&y_2^{(4)}(0.5x-0.3)+2y_1^{(3)}(0.5x-0.3)\\
&-xy_3^{(2)}(0.5x-0.3)+y_2(0.5x-0.3)\\
&= -x\mathrm{e}^{\left(\frac{1}{2}x-\frac{3}{10}\right)},\\
&\mathrm{e}^x y_3^{(4)}(0.5x-0.3)+y_2^{(2)}(0.5x-0.3)\\
&-y_1^{(1)}(0.5x-0.3)-\mathrm{e}^x y_3(0.5x-0.3)\\
&= -2\cos\left(\frac{1}{2}x-\frac{3}{10}\right),\\
&0 \le x \le 1,
\end{aligned}
\tag{27}
$$

with the initial conditions $y_1(0)=0$, $y_1^{(1)}(0)=1$, $y_1^{(2)}(0)=0$, $y_1^{(3)}(0)=-1$, $y_2(0)=1$, $y_2^{(1)}(0)=0$, $y_2^{(2)}(0)=-1$, $y_2^{(3)}(0)=0$, $y_3(0)=y_3^{(1)}(0)=y_3^{(2)}(0)=$

Table 5. Numerical results of the absolute error functions of $y_1(x)$ of (26).

x_i	Chebyshev method [19] $e_{1,5}(x_i)$	Absolute errors with Stehfest method [27]	Present method $e_{1,5}(x_i)$	$e_{1,7}(x_i)$	$e_{1,10}(x_i)$
0.1	4.510522e-005	6.7614e-005	5.9187e-007	1.3161e-009	5.1070e-014
0.2	7.985043e-005	8.4949e-005	1.0110e-006	1.4770e-009	5.0182e-014
0.5	9.719089e-005	3.18972e-003	1.0978e-006	1.8584e-009	7.1942e-014
0.8	8.006002e-005	5.20283e-003	9.0094e-007	2.4772e-009	5.1514e-014
1	1.067677e-004	1.193776e-002	1.3659e-005	2.7917e-009	1.6289e-012

Table 6. Numerical results of the absolute error functions of $y_2(x)$ of (26).

x_i	Chebyshev method [19] $e_{2,5}(x_i)$	Absolute errors with Stehfest method [27]	Present method $e_{2,5}(x_i)$	$e_{2,7}(x_i)$	$e_{2,10}(x_i)$
0.1	2.247723e-005	8.4086e-006	2.9327e-007	6.5736e-010	2.8644e-014
0.2	3.984701e-005	1.9575e-005	5.0134e-007	7.3785e-010	3.9524e-014
0.5	4.890662e-005	2.242e-004	5.4596e-007	9.2883e-010	1.3589e-013
0.8	4.064222e-005	4.647e-004	4.5116e-007	1.0741e-009	3.0687e-013
1	5.390356e-005	4.710e-004	6.7555e-006	1.6585e-008	3.4728e-013

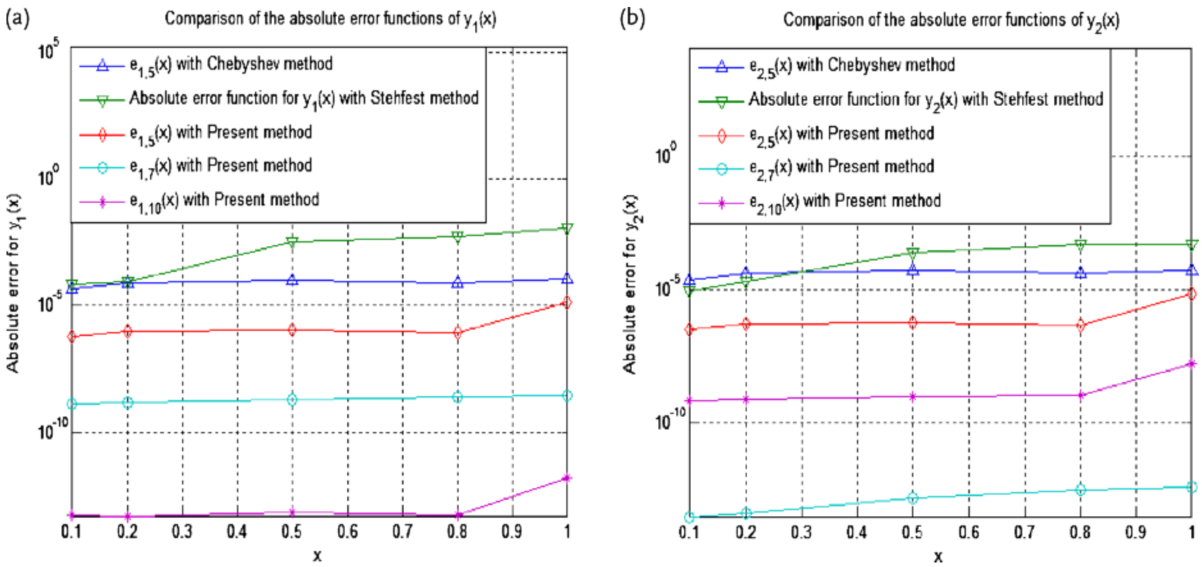

Fig. 2 (colour online). For $N = 5$, $N = 7$, and $N = 10$ of (26), (a) comparison of the absolute errors functions $e_{i,N}(x)$ for $y_{1,N}(x)$, and (b) comparison of the absolute errors functions $e_{i,N}(x)$ for $y_{2,N}(x)$.

Table 7. Numerical results of solution $y_1(x)$ of (27).

x_i	Exact solution $y_1(x_i)$	Present method $N=5$, $y_{1,5}(x_i)$	$N=9$, $y_{1,9}(x_i)$	$N=12$, $y_{1,12}(x_i)$
0	0	0	0	0
0.2	0.198669330795061	0.198668543502482	0.198669330794336	0.198669330795058
0.4	0.389418342308651	0.389404343958251	0.389418342232165	0.389418342308598
0.6	0.564642473395035	0.564566958876156	0.564642471881110	0.564642473392608
0.8	0.717356090899523	0.717114236724616	0.717356077692792	0.717356090840826
1	0.841470984807896	0.840909676320474	0.841470916589261	0.841470984048470

Table 8. Numerical results of solution $y_2(x)$ of (27).

x_i	Exact solution $y_2(x_i)$	Present method $N=5$, $y_{2,5}(x_i)$	$N=9$, $y_{2,9}(x_i)$	$N=12$, $y_{2,12}(x_i)$
0	1	1	1	1
0.2	0.980066577841242	0.980069204134346	0.980066577846321	0.980066577841242
0.4	0.921060994002885	0.921118059446279	0.921060994648016	0.921060994002882
0.6	0.825335614909678	0.825714817011519	0.825335630365316	0.825335614909456
0.8	0.696706709347165	0.698234336636051	0.696706877673934	0.696706709342594
1	0.540302305868140	0.544939036581650	0.540303452419596	0.540302305821106

Table 9. Numerical results of solution $y_3(x)$ of (27).

x_i	Exact solution $y_3(x_i)$	Present method $N=5$, $y_{3,5}(x_i)$	$N=9$, $y_{3,9}(x_i)$	$N=12$, $y_{3,12}(x_i)$
0	1	1	1	1
0.2	1.22140275816017	1.22140039799556	1.22140275815564	1.22140275816017
0.4	1.49182469764127	1.49177302875795	1.49182469705621	1.49182469764126
0.6	1.82211880039051	1.82176967853381	1.82211878612799	1.82211880038835
0.8	2.22554092849247	2.22410160665451	2.22554077023538	2.22554092843659
1	2.71828182845905	2.71378950175349	2.71828072901644	2.71828182772375

Table 10. Numerical results of the absolute error functions $e_{i,N}(x)$ for $i=1,2,3$ and $N=5,9,12$ of (27).

	Present method								
	The absolute errors for $y_1(x)$			The absolute errors for $y_2(x)$			The absolute errors for $y_3(x)$		
x_i	$e_{1,5}(x_i)$	$e_{1,9}(x_i)$	$e_{1,12}(x_i)$	$e_{2,5}(x_i)$	$e_{2,9}(x_i)$	$e_{2,12}(x_i)$	$e_{3,5}(x_i)$	$e_{3,9}(x_i)$	$e_{3,12}(x_i)$
0	0	0	0	0	0	0	0	0	0
0.2	7.8729e-007	7.2561e-013	2.9976e-015	2.6263e-006	5.0796e-012	1.1102e-016	2.3602e-006	4.5342e-012	2.4425e-015
0.4	1.3998e-005	7.6485e-011	5.2625e-014	5.7065e-005	6.4513e-010	2.7756e-015	5.1669e-005	5.8506e-010	7.3275e-015
0.6	7.5515e-005	1.5139e-009	2.4271e-012	3.7920e-004	1.5456e-008	2.2204e-013	3.4912e-004	1.4263e-008	2.1592e-012
0.8	2.4185e-004	1.3207e-008	5.8697e-011	1.5276e-003	1.6833e-007	4.5712e-012	1.4393e-003	1.5826e-007	5.5876e-011
1	5.6131e-004	6.8219e-008	7.5943e-010	4.6367e-003	1.1466e-006	4.7033e-011	4.4923e-003	1.0994e-006	7.3530e-010

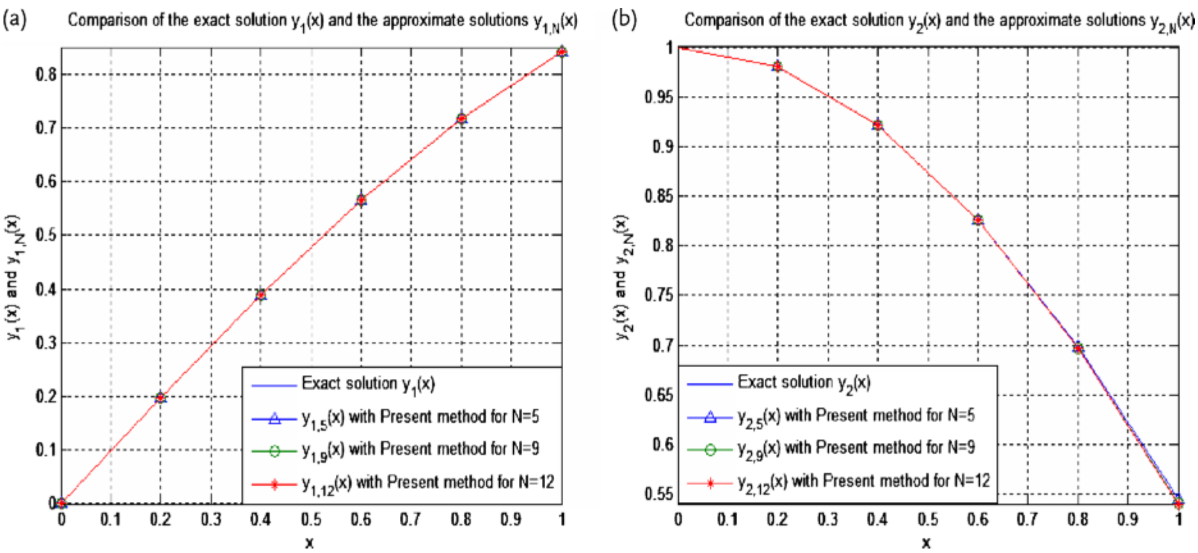

Fig. 3 (colour online). For $N=5$, $N=9$, and $N=12$ of (27), (a) comparison of the exact solution $y_1(x)$ and the approximate solutions $y_{1,N}(x)$, (b) comparison of the exact solution $y_2(x)$ and the approximate solutions $y_{2,N}(x)$, (c) comparison of the exact solution $y_3(x)$ and the approximate solutions $y_{3,N}(x)$, (d) comparison of the absolute errors functions $e_{1,N}(x)$ for $y_{1,N}(x)$ in the interval $0.2 \le x \le 1$, (e) comparison of the absolute errors functions $e_{2,N}(x)$ for $y_{2,N}(x)$ in the interval $0.2 \le x \le 1$, (f) comparison of the absolute errors functions $e_{3,N}(x)$ for $y_{3,N}(x)$, in the interval $0.2 \le x \le 1$.

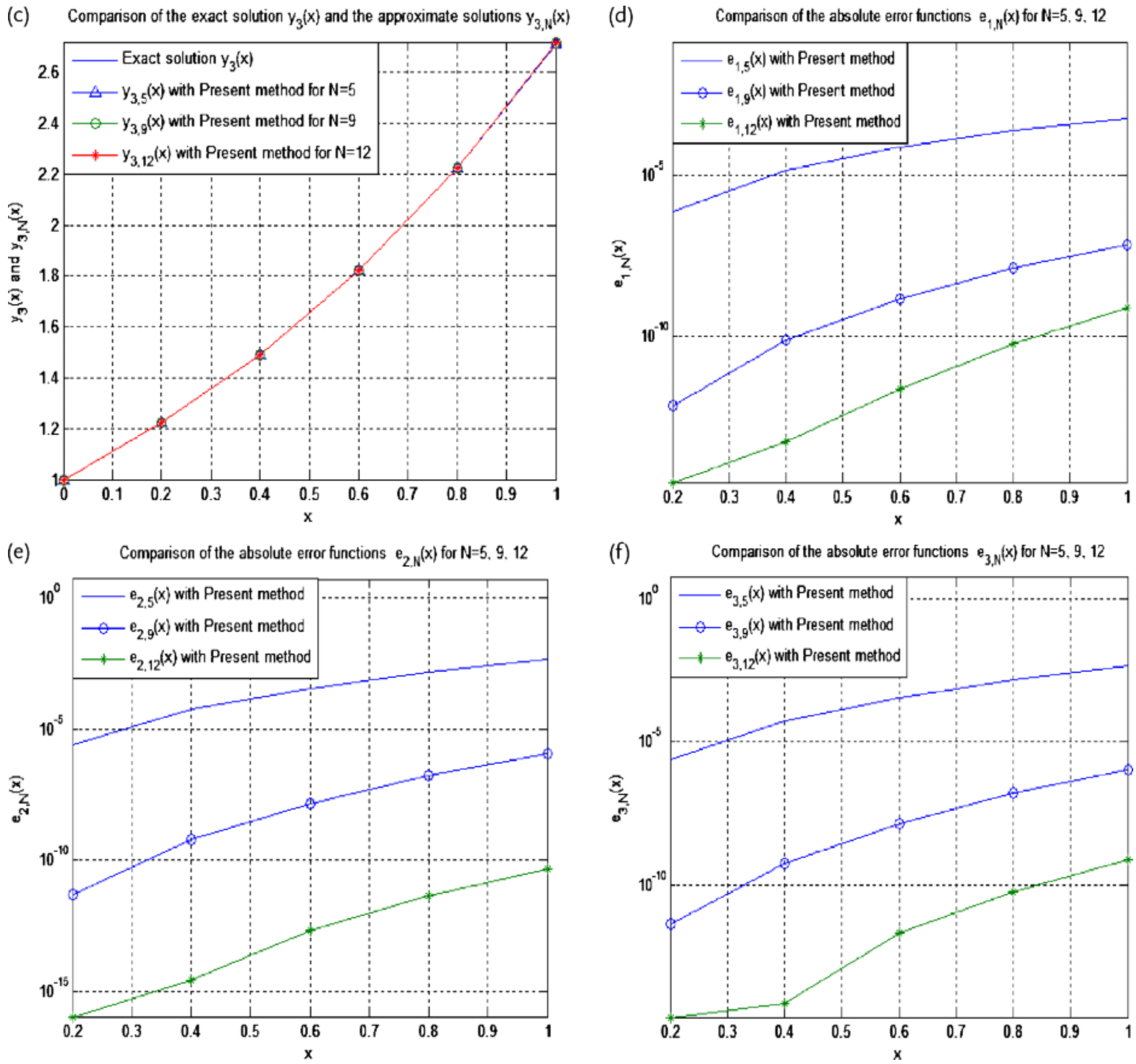

Fig. 3 (colour online). Continued.

$y_3^{(2)}(0) = 1$, and the exact solutions $y_1(x) = \sin(x)$, $y_2(x) = \cos(x)$, $y_3(x) = e^x$.

From (15), the fundamental matrix equation of this system is

$$\{\,\mathbf{P}_0\mathbf{X}\tilde{\mathbf{B}}\bar{\mathbf{D}} + \mathbf{P}_1\mathbf{X}\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^1\bar{\mathbf{D}} + \mathbf{P}_2\mathbf{X}\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^2\bar{\mathbf{D}} + \mathbf{P}_3\mathbf{X}\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^3\bar{\mathbf{D}} + \mathbf{P}_4\mathbf{X}\tilde{\mathbf{B}}\left(\bar{\mathbf{B}}\right)^4\bar{\mathbf{D}}\,\}\mathbf{A} = \mathbf{G}.$$

Thereby, taking $N = 5, 9, 12$, we gain the approximate solution by Bessel polynomials of this system. Tables 7 – 10 and Figure 3 show numerical results of the approximate solutions and the absolute error functions obtained by the present method for various N and the exact solution of (27). It seems that the accuracy increases as N is increased. From Tables 7 – 10 and Figure 3, the obtained results have shown speedy convergence.

6. Conclusions

High-order systems of linear differential-difference equations are usually difficult to solve analytically. In many cases, it is required to approximate solutions. In this article, a new technique, using the Bessel polynomials of the first kind, to numerically solve systems of high-order linear differential-difference equations is presented. The comparision of the results obtained by the present method, the exact solutions, and the other methods reveals that the present method is very effective and convenient. One of the considerable advantage of the method is that the approximate solutions are found very easily by using the computer code written in MATLAB v7.6.0 (R2008a). Shorter computation time and lower operation count results in a reduction of cumulative truncation errors and an improvement of overall accuracy.

The method can be developed and applied to systems of linear integral and integro-differential-difference equations but some modifications are required.

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